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Existence of Proper Modes of Helicon Oscillations*

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In this paper it is shown that the class of electromagnetic problems for which the operator $i(\partial/\partial t)$ (where t denotes time) is self-adjoint extends beyond problems involving only insulators and perfect conductors. The class includes problems in which the perfect conductor is generalized to a medium with antisymmetric resistivity tensor. The latter medium approximates media in which *helicon* waves can propagate. Helicon waves are known to propagate in good conductors in a strong magnetic field $\mathbf{B}_0$; it will be found that two necessary conditions for self-adjointness of the operator $i(\partial/\partial t)$ are that the sample carrying helicons must not have a finite portion parallel to $\mathbf{B}_0$, and it must be surrounded by a reflecting surface that prevents energy from escaping.

1. INTRODUCTION

HELICONS are circularly polarized electromagnetic waves, which can propagate almost without attenuation inside a solid of high conductivity, permeated by a strong magnetic field. (The condition for unattenuated propagation is that $\omega_c\tau \gg 1$, where ω_c is the cyclotron frequency of the current carriers and τ is the relaxation time.) In atmospheric physics these waves are well known as *radio whistlers*.[1] The term "helicon," which is presently accepted to denote whistlers in the context of solid state physics, is due to Aigrain,[2] who first proposed achievable experiments to detect them in solids.[3] Observation of helicons was first announced by Bowers, Legéndy, and Rose,[4] at frequencies of order 10 cps. In recent years, helicons have been observed and studied by a number of authors; a short survey of the literature on the subject is given in another article by the present author.[5]

In this article we deal with the abstract boundary-value problem presented by helicons under three idealizing assumptions: (i) The sample carrying helicons has negligible resistivity (but is not a superconductor, so that it does not exclude magnetic flux). (ii) The resistivity tensor is the same as would be for uniform dc fields. (That is, nonlocal effects in space and time are ignored.) (iii) The constitutive equation is linearized. Assumptions (i) to (iii) amount to assuming that (owing to the presence of the external magnetic field mentioned above) the sample carrying helicons is characterized by a fixed, antisymmetric resistivity tensor.

The arrangement we consider consists of a sample (region M_1 in Fig. 1) and a closed reflecting surface * (surface S in Fig. 1) surrounding it, to stop any energy from escaping; between the sample and the reflecting surface there is a nonconducting region (region M_2 in Fig. 1). The net charge on the sample, and the charge density in the nonconducting region are assumed to be zero. The sample is required to have smooth boundaries; a further requirement on the boundaries is that they have no finite portion parallel to the external magnetic field.

Under the above assumptions, the operator $-i(\partial/\partial t)$, operating on electromagnetic fields, is shown to be self-adjoint. In the proof it is not necessary to assume that the dielectric constant, magnetic susceptibility, and Hall coefficient are constants throughout the regions of interest; the external magnetic field is not required to be uniform, nor the displacement current negligible. Aside from the restrictions already stated, there is no restriction on the shape of the sample; no use is made of any assumptions to the effect that the sample is connected or simply connected.

The purpose of making the seemingly arbitrary restriction, that the boundary shall have no finite portion tangential to the external magnetic field, is to avoid a certain surface mode of energy absorption[5,6] confined to surfaces tangential to the external field. If the resistivity is assumed to be finite, and is then allowed to tend to zero, the electric currents in this mode increase without bound, and the Ohmic loss does not tend to zero. Therefore, in samples with such surfaces, any free oscillations are bound to be attenuated, the operator $-i(\partial/\partial t)$ cannot have real eigenvalues, and cannot be self-adjoint. (If anoma-

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[2] P. Aigrain, Proc. Intern. Conf. Semicond. Phys., Prague, **1960**, 224 (1961).
[3] See, however, O. V. Konstantinov and V. I. Perel', Zh. Eksperim. i Teor. Fiz. **38**, 161 (1960) [English transl.: Soviet Phys.—JETP **11**, 117 (1960)].
[4] R. Bowers, C. R. Legéndy, and F. E. Rose, Phys. Rev. Letters **7**, 339 (1961).
[5] C. R. Legéndy, Phys. Rev. **135**, A 1713 (1964).
[6] J. M. Goodman and C. R. Legéndy (unpublished).
*Note added in the Internet edition – see end of article

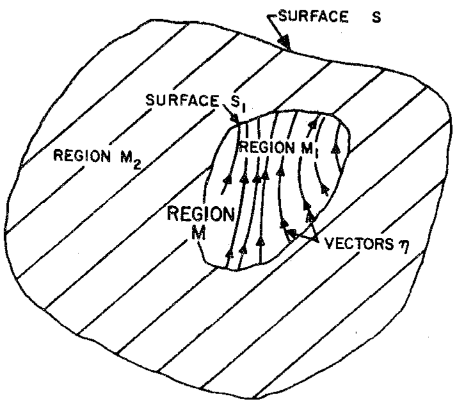

FIG. 1. Notation for Sec. 2. Region M consists of regions M_1 and M_2. M_1 is the sample carrying helicons.

lous skin effect is taken into consideration, the surface loss is found to disappear for low enough resistivity.[5])

Electromagnetic fields will be represented as vectors [see (2.1)] in an abstract vector space with a scalar product [see (2.2)], as was done by Marcuwitz[7] and Wilcox.[8] The present formulation slightly differs from theirs, in that, throughout Sec. 2 we deal only with *instantaneous* electromagnetic fields, and do not even implicitly assume any time dependence [such as exp $(i\omega t)$]. The operator $\hat{T} = -i(\partial/\partial t)$ is rewritten, using two of Maxwell's equations, so that it operates on the spatial coordinates only [see Eq. (2.3)]. None of Maxwell's equations are explicitly used in defining the allowable electromagnetic fields; instead, it is required that the instantaneous field be in the *range* of the operator $\hat{T}$. The two time-independent Maxwell's equations are then automatically satisfied (because of the previously stated assumptions of charge neutrality). After the self-adjointness of $\hat{T}$ is established (and therefore, the existence of a complete set of eigenfunctions is shown), time dependence is introduced by means of the operator exp $(it\hat{T})$. The resulting time-dependent functions automatically satisfy the two time-dependent Maxwell's equations.

In Sec. 2 we give the mathematical definitions and proofs, then make the necessary physical connections in Sec. 3.

2. DEFINITIONS, THEOREM, AND PROOF

Definitions. Let S be a smooth, simply connected, closed surface; M the region composed of all points (x, y, z), inside and on S; S_1 a smooth, closed surface entirely surrounded by S (and not touching S); M_1 the region composed of points inside and on S_1, and let M_2 be the rest of M (see Fig. 1). Define the vector $\mathfrak{y}$ in the region M_1, as an everywhere-bounded, real, and well-behaved function of x, y, z, with the further restriction that on the surface S_1, the scalar product $\mathfrak{y}\cdot\mathbf{n} \neq 0$ (where $\mathbf{n}$ is a normal vector to S_1), except at most on some isolated points or lines. Let $\mathbf{E}(x, y, z)$ and $\mathbf{H}(x, y, z)$ be (possibly complex) vector functions, both defined throughout M such that $\nabla\times\mathbf{E}$ and $\nabla\times\mathbf{H}$ are well defined; let $\epsilon(x, y, z)$ and $\mu(x, y, z)$ be everywhere positive, real and bounded functions, defined throughout M.

Form the six-component vectors

$$\mathsf{F} = \begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix}; \tag{2.1}$$

for any two such vectors, define the scalar product:

$$(\mathsf{F}_1, \mathsf{F}_2) = \frac{1}{2}\int_{\text{over } M} (\epsilon\mathbf{E}_1^*\cdot\mathbf{E}_2 + \mu\mathbf{H}_1^*\cdot\mathbf{H}_2)\, dV, \tag{2.2}$$

and on all such vectors define the operator $\hat{T}$ as follows:

$$\hat{T}\mathsf{F} = \begin{pmatrix}-i\epsilon^{-1}(\nabla\times\mathbf{H} - \mathbf{j})\\ i\mu^{-1}\nabla\times\mathbf{E}\end{pmatrix}, \tag{2.3a}$$

where

$$\mathbf{j} = \begin{cases}\eta^{-2}[\mathfrak{y}(\mathfrak{y}\cdot\nabla\times\mathbf{H}) - \mathbf{E}\times\mathfrak{y}] & \text{in } M_1\\ 0 \quad \text{in } M_2\end{cases}$$
$$\eta = (\mathfrak{y}_x^2 + \mathfrak{y}_y^2 + \mathfrak{y}_z^2)^{1/2}. \tag{2.3b}$$

Let the domain $\mathfrak{D}$ of $\hat{T}$ be the set of F satisfying the following boundary conditions (almost) everywhere on S and S_1 respectively:

$$(\alpha\mathbf{E} + \beta\mathbf{H})\times\mathbf{n} = 0 \quad \text{on} \quad S \tag{2.4a}$$

$$\left.\begin{matrix}\mathbf{E}\times\mathbf{n}\\ \mathbf{H}\times\mathbf{n}\end{matrix}\right\} \quad \text{continuous across } S_1 \tag{2.4b}$$

where α and β are fixed, real, scalar functions of the position on S; both differentiable once, and at least one of them differing from zero at each point on S.

Let L^2 be the space of all vectors F for which $(\mathsf{F}, \mathsf{F}) < \infty$. Let $\mathcal{P}$ be the closure of the range of $\hat{T}$. One can show at once that $\mathcal{P}$ is the set of vectors F for which

$$\nabla\cdot\mu\mathbf{H} = 0 \quad \text{throughout } M,$$
$$\mathbf{E}\cdot\mathfrak{y} = 0 \quad \text{in} \quad M_1, \tag{2.5}$$
$$\nabla\cdot\epsilon\mathbf{E} = 0 \quad \text{in} \quad M_2,$$
$$\oint_{S_1} \epsilon\mathbf{E}\cdot d\mathbf{S} = 0,$$

[7] N. Marcuwitz, *Electromagnetic Waves*, Proceedings of a Symposium Conducted by the Mathematics Research Center, U. S. Army, at the University of Wisconsin, Madison, on 10–12 April 1961; edited by R. E. Langer (The University of Wisconsin Press, Madison, 1962), p. 109.

[8] C. H. Wilcox, Ref. 7, p. 65.

where S_2 is any closed surface in the region M_2, enclosing M_1.

Define the Hilbert space $\mathfrak{F}$ as the set of all L^2 vectors in $\mathcal{P}$.

Lemma. Defined on the domain $\mathfrak{D}$, the operator $\hat{T}$ is symmetric, i.e., if F_1 and F_2 are both in $\mathfrak{D}$, then

$$(\mathsf{F}_1, \hat{T}\mathsf{F}_2) = (\hat{T}\mathsf{F}_1, \mathsf{F}_2).$$

Proof. It is enough to show that for all F in $\mathfrak{D}$,

$$\Delta = \frac{i}{2}[(\mathsf{F}, \hat{T}\mathsf{F}) - (\hat{T}\mathsf{F}, \mathsf{F})] = \mathrm{Re}\,(\mathsf{F}, i\hat{T}\mathsf{F}) = 0. \quad (2.6)$$

For then the substitution $\mathsf{F} = \mathsf{F}_1 + i\mathsf{F}_2$ in (2.6) shows that the real part of $(\mathsf{F}_1, \hat{T}\mathsf{F}_2) - (\hat{T}\mathsf{F}_1, \mathsf{F}_2)$ vanishes, and the substitution $\mathsf{F} = \mathsf{F}_1 + \mathsf{F}_2$ shows that the imaginary part of the same expression vanishes for all F_1 and F_2 in $\mathfrak{D}$.

To prove (2.6), integrate (2.2) by parts, thus splitting up the integral into surface and volume integrals:

$$\Delta = \int_{M_1} A + \int_{M_2} A + \oint_{S_1^{(1)}} B - \oint_{S_1^{(2)}} B + \oint_S B$$

where

$$A = \tfrac{1}{2}\,\mathrm{Re}\,(\mathbf{E}^*\cdot\mathbf{j})\,dV$$

$$B = -\tfrac{1}{2}\,\mathrm{Re}\,(\mathbf{E}^*\times\mathbf{H})\cdot d\mathbf{S}$$

and $S_1^{(1)}$ and $S_1^{(2)}$ refer to integrals over S_1 as the surface is approached from region M_1 and region M_2 respectively. From the definition of $\mathbf{j}$ in (2.3) one can see at once that A identically vanishes in M_1 as well as M_2, thus, the first two integrals vanish. (It is at this point that the antisymmetry of the resistivity tensor was exploited.) Because of the boundary condition on S_1, (2.4b), the two surface integrals over S_1 cancel. Finally, from the boundary condition over S, (2.4a), the last integral vanishes, which completes the proof of the *lemma.*

Theorem. In the space $\mathfrak{F}$ and on the domain $\mathfrak{D}$, the operator $\hat{T}$ is self-adjoint.

Proof. In view of the *lemma* just proved, it is enough to show that there exist a set of vectors $\{\mathsf{F}_i\}$ that are in $\mathfrak{F}$ as well as $\mathfrak{D}$, and are such that if for some F,

$$(\mathsf{F}, \hat{T}\mathsf{F}_i) = (\hat{T}\mathsf{F}, \mathsf{F}_i) \quad (2.7)$$

for all F_i in the set, then F is necessarily in the domain $\mathfrak{D}$.

To show this, first form two arbitrary, complete sets of everywhere bounded and differentiable vector functions $\{\mathbf{a}_i\}$ and $\{\mathbf{b}_i\}$, defined only on the points of S_1, everywhere tangential to S_1 and $\mathbf{b}_i$ identically vanishing wherever $\mathfrak{y}\cdot\mathbf{n} = 0$. Completeness is meant in the sense that, if the vector functions $\mathbf{P}(x, y, z)$ and $\mathbf{Q}(x, y, z)$, defined throughout M are such that

$$\oint_{S_1} (\mathbf{P}\cdot\mathbf{a}_i)\,dS = 0, \quad \oint_{S_1} (\mathbf{Q}\cdot\mathbf{b}_i)\,dS = 0 \quad (2.8)$$

for all i, then (almost) everywhere on S_1,

$$\mathbf{P}\times\mathbf{n} = 0, \qquad \mathbf{Q}\times\mathbf{n} = 0. \quad (2.9)$$

(The latter of these can only be true because, by definition, $\mathfrak{y}\cdot\mathbf{n} \neq 0$ almost everywhere on S_1.) From a_i and b_i construct the six-component vectors

$$\mathsf{G}_i = \begin{pmatrix} 0 \\ \mathbf{p}_i \end{pmatrix}, \qquad \mathsf{K}_i = \begin{pmatrix} \mathbf{q}_i \\ 0 \end{pmatrix}$$

such that $\mathbf{p}_i$ and $\mathbf{q}_i$ identically vanish everywhere on and near S, they are bounded and well-behaved everywhere, and

$$\mathbf{p}_i\times\mathbf{n} = \mathbf{a}_i; \qquad \mathbf{q}_i\times\mathbf{n} = \mathbf{b}_i;$$

on S_1. (Note that, necessarily, G_i and K_i are in the domain $\mathfrak{D}$.) If $\mathbf{p}_i$ and $\mathbf{q}_i$ satisfy the above requirements, G_i and K_i are in the space $\mathfrak{F}$ if, and only if

$$\begin{aligned} \nabla\cdot\mu\mathbf{p}_i &= 0 \quad \text{throught } M, \\ \mathbf{q}_i\cdot\mathfrak{y} &= 0 \quad \text{in} \quad M_1, \\ \nabla\cdot\epsilon\mathbf{q}_i &= 0 \quad \text{in} \quad M_2, \\ \oint_{S_2} \epsilon\mathbf{q}_i\cdot d\mathbf{S} &= 0, \end{aligned} \quad (2.10)$$

where S_2 is any surface enclosing M_1 [Cf. Eq. (2.5)]. Under the assumptions made earlier, the conditions listed [including (2.10)] are not very restrictive, and there is a wide choice of $\mathbf{p}_i$ and $\mathbf{q}_i$ satisfying them. [However, if for any i we had $\mathbf{b}_i\times\mathfrak{y} \neq 0$ at a point where $\mathfrak{y}\cdot\mathbf{n} = 0$, there would exist no $\mathbf{q}_i$ satisfying the second of Eqs. (2.10).]

Substitute for F_i in Eq. (2.7) the vectors G_i and K_i; it follows at once from (2.8) and (2.9) that F in Eq. (2.7) satisfies the boundary condition on S_1.

To carry out the analogous proof for S, form a complete set of vector functions $\mathbf{c}_i$, defined on the points of S, with properties similar to the properties of $\mathbf{a}_i$ previously defined on S_1. Then form

$$\mathsf{L}_i = \begin{pmatrix} \mathbf{e}_i \\ \mathbf{h}_i \end{pmatrix}$$

such that $\mathbf{e}_i$ and $\mathbf{h}_i$ identically vanish in and near M_1, they are bounded and well behaved everywhere, and

$$\mathbf{e}_i\times\mathbf{n} = \beta\mathbf{c}_i;\, \mathbf{h}_i\times\mathbf{n} = -\alpha\mathbf{c}_i,$$

on S [where α and β are as defined in Eq. (2.4a)]. The vector L_i thus defined is necessarily in the domain $\mathfrak{D}$. It is easy to see that if L_i satisfy the above requirements, they are in $\mathfrak{F}$ if, and only if $\nabla \cdot \epsilon \mathbf{e}_i = 0$ and $\nabla \cdot \mu \mathbf{h}_i = 0$ everywhere. Again, the conditions are not very restrictive, and there is a wide choice of L_i satisfying them.

Substituting the vectors L_i for F_i in Eq. (2.7), and using the completeness of the set $\{c_i\}$ through observations such as (2.8) and (2.9), it is easily shown that F in Eq. (2.7) satisfies the boundary condition on S, and the proof is complete.

3. RESULTS AND DISCUSSION

The vectors $\mathbf{E}$ and $\mathbf{H}$ in (2.1) are recognized as the electric and magnetic field; the scalar product of a vector by itself, (F, F), is recognized as the energy in the electromagnetic field F. As was indicated in the Introduction, the operator $\hat{T}$, defined in (2.3) is identified at once as $-i(\partial/\partial t)$ [$\mathbf{j}$ in (2.3) standing for electric current density]. The definition of $\mathbf{j}$ in the region M_1, the region carrying helicons, is so designed as to make $\mathbf{E} = \mathbf{j} \times \mathfrak{y}$, and therefore $\mathfrak{y} \cdot \mathbf{E} = 0$ and $\mathfrak{y} \cdot (\partial \mathbf{E}/\partial t) = 0$ at all times. Physically, $\mathfrak{y} = -R\mathbf{B}_0$, where R is the Hall coefficient and $\mathbf{B}_0$ is the steady, external magnetic field. The boundary condition (2.4a) forces Poynting's vector $\mathbf{E} \times \mathbf{H}$ to be tangential to S, hence the surface S reflects all radiation coming onto it.

Denote the integrand in (2.2) as $\mathsf{F}^*\mathsf{F}$. Then

$$2\,\mathrm{Re}\,(\mathsf{F}, i\hat{T}\mathsf{F}) = \frac{1}{2}\int_M \left(\mathsf{F}^* \frac{\partial \mathsf{F}}{\partial t} + \frac{\partial \mathsf{F}^*}{\partial t}\mathsf{F}\right) = \frac{\partial}{\partial t}(\mathsf{F}, \mathsf{F}). \tag{3.1}$$

Comparing (3.1) with (2.6), it is found that the operator $\hat{T}$ is symmetric if and only if the system conserves energy. We recall from the proof of the *lemma* that symmetry hinged upon three facts: (1) in region M_1 the resistivity tensor is antisymmetric, therefore the current and electric field are perpendicular, and there is no Ohmic loss; (2) by (2.4b), the normal component of Poynting's vector is continuous across S_1; and (3) by (2.4a), the normal component of Poynting's vector is zero over S. The mathematical fact that a symmetric operator has real eigenvalues is translated into the statement that a system conserving energy cannot execute damped or growing oscillations. The mathematical fact that a symmetric operator has orthogonal eigenvectors corresponds to the statement that if the electromagnetic system conserves energy, its total energy is the sum of the energies in the individual modes.

It follows from a remark made below (2.10) that if over a finite portion of the sample's surface $\mathfrak{y} \cdot \mathbf{n} = 0$, it is not possible to establish that F in (2.7) satisfies the boundary condition (2.4b). Indeed, physical considerations[5] show that in that case, under the assumptions on which the present formulation is built [namely, Assumptions (i) to (iii) in the Introduction], the boundary of the sample absorbs energy, and the proof cannot be completed. (However, there is no difficulty in carrying out the proof if the surfaces in question are appropriately tilted or are made slightly "wavy"; for the purposes of the proof it does not matter how slight the distortion is. The artificially introduced roughness of the surface may be thought of as "simulating" anomalous skin effect, in that it eliminates surface loss for low enough resistivities. For this simulation to fit the physical situation best, the depth of roughness must be of the same order of magnitude as the cyclotron radius.)

It is proved in functional analysis[9] that if an operator $\hat{T}$ is self-adjoint in a Hilbert space $\mathfrak{F}$, then the equation

$$\hat{T}\mathsf{F} = \omega \mathsf{F} \tag{3.3}$$

possesses a set of eigenfunctions F which span all of $\mathfrak{F}$. The eigenfunctions are orthogonal and the eigenvalues ω are real. A glance at the definition (2.3) of $\hat{T}$ shows that the two time-dependent Maxwell's equations can be compressed into the form

$$-i\,\partial \mathsf{F}(t)/\partial t = \hat{T}\mathsf{F}(t). \tag{3.4}$$

It follows that once the self-adjointness of $\hat{T}$ is established, it is possible to construct a time-dependent field $\mathsf{F}(t)$ from any instantaneous field F in $\mathfrak{F}$ as follows:

$$\mathsf{F}(t) = e^{it\hat{T}}\mathsf{F}. \tag{3.5}$$

The field (3.5) thus constructed satisfies Maxwell's equations (3.4). The sequence of expressions (3.4), (3.3), (3.5) resembles the sequence of expressions encountered in connection with Schrödinger's equation, with similar causal relations between the successive forms.

The self-adjointness of $\hat{T}$ implies that Eq. (3.5) is meaningful, but it does not imply that the eigenfunctions of (3.3) have finite energy, i.e., that the eigenfrequencies ω form a discrete set. It is hoped that in the near future some author will show that $\hat{T}$ has a unique inverse, and that the inverse is com-

[9] F. Riesz and B. Sz.-Nagy, *Functional Analysis* (Frederick Ungar Publishing Company, New York, 1955).

pletely continuous. This, together with the symmetry property proved in the *lemma*, would imply everything implied by self-adjointness, and would also imply discreteness and square integrability.

We remark that the proofs of Sec. 2 do not make use of the fact that the space $\mathfrak{F}$ is restricted to the range of $\hat{T}$. Both proofs can be repeated without difficulty if the first, third, and fourth of Eqs. (2.5) are dropped, as long as the second is retained and the domain $\mathfrak{D}$ is defined by (2.4). Of course, the results are then not physically meaningful. Also, if $\hat{T}$ has a larger domain than range, it cannot possibly turn out to have an inverse, as was suggested in the previous paragraph.

Eqs. (2.5) can be compressed into the statement that F must be in the closure of the range of $\hat{T}$. The physical interpretation of these equations is clear. The first one is Maxwell's equation; the third is also Maxwell's equation, assuming that region M_2 contains no free charges; the fourth requires that there be no net charge on the sample; the second, combined with (2.3b) means that the resistivity tensor is antisymmetric.

It is a feature of the present formulation that all field equations, more precisely, the four Maxwell's equations and the constitutive equation, are introduced into the problem merely through the definition of a single operator.

In closing, we wish to comment on the reflecting surface S. The operator $\hat{T}^2$ has positive eigenvalues, therefore, if φ is any vector in $\mathfrak{F}$, the quantity

$$(\varphi, \hat{T}^2\varphi)/(\varphi, \varphi) \tag{3.6}$$

is larger than the square of the smallest eigenfrequency. Thus, to make a crude estimate, let

$$\varphi = \begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix}, \qquad \mathbf{E} = (\nabla \times \mathbf{H}) \times \mathfrak{y}, \qquad \mathbf{H} = \mu_0^{-1}\nabla \times \mathbf{A},$$

and

$$A_x = A_y = 0,$$

$$A_z = \begin{cases} \left(\cos \dfrac{\pi x}{l}\right)^4 \left(\cos \dfrac{\pi y}{l}\right)^4 \left(\cos \dfrac{\pi z}{l}\right)^4 \\ \qquad \text{for } |x| \le \dfrac{l}{2}, \quad |y| \le \dfrac{l}{2}, \quad |z| \le \dfrac{l}{2} \\ 0 \quad \text{otherwise;} \end{cases}$$

choose the origin of the coordinate system in such a way that the cubical region inside which $\varphi \neq 0$ be in the interior of the sample, and choose the edge of the cube, l, to be as large as possible. Substitute the the resulting field φ into (3.6), and assume, for simplicity, that $\epsilon \equiv \epsilon_0$, $\mu \equiv \mu_0$ and $\mathfrak{y}$ is uniform. Then,

$$[(\varphi, \hat{T}^2\varphi)/(\varphi, \varphi)]^{\frac{1}{2}} \sim 14(2\pi/l)^2\mathfrak{y}\mu^{-1}.$$

Thus, the smallest eigenfrequency is necessarily smaller than the latter quantity. Since we chose φ such that $\varphi \equiv 0$ outside the sample, the above estimate only depends on the sample's size, and not on the dimensions of the reflecting surface. The estimate shows that, if the wavelength in free space corresponding to the lowest mode is denoted by λ_0, then, to order of magnitude, $\lambda_0/l \sim (l/2\pi)(\mu_0/\epsilon_0)^{\frac{1}{2}}\mathfrak{y}^{-1}$. In the physical situation of Ref. 4 (but not in the situation of Ref. 2), $(l/2\pi)(\mu_0/\epsilon_0)^{\frac{1}{2}}\mathfrak{y}^{-1} \sim 10^8$, i.e., independently of the size of the reflecting shield, the lowest modes can be considered quasistatic (i.e., of essentially infinite vacuum wavelength). To estimate the rate at which energy would leave the region M_1 in the absence of the reflecting surface, consider the fields due to the currents and charges on the sample alone; neglect all but the magnetic dipole radiation, and let the shield be a sphere of radius λ_0. The ratio of the energy crossing the shield in one cycle to the energy inside the shield is then found to be of order $(l/\lambda_0)^3 \sim 10^{-24}$.

For the higher modes the rate of radiation is higher. However, it can be shown that if we formally let the speed of light outside the sample tend to infinity, the set of almost unattenuated modes can be extended to an arbitrarily large part of the spectrum.

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Footnote to page 153 (added in the Internet edition)

At the frequencies of interest in Reference 4 the reflecting surface is not necessary, since an effect similar to total internal reflection confines the helicon energy to the sample, and essentially no energy reaches the reflecting shield. As seen in the last paragraphs of the article ("comment on the reflecting surface S"), at the frequencies in question the fraction of energy reaching the shield in one cycle would be of order 10^{-24} (meaning that it would take some 10^{14} years before most of the energy leaked out). Nevertheless, the rigorous proof fails without the shield, because to the mathematics a small leakage of energy is no different from a substantial one, and the energy leakage at the very high eigenfrequencies (where leakage is substantial) is as important as the leakage at the low eigenfrequencies (where it is not). Accordingly the proof of this paper has been split into a "mathematical proof" and a "physical proof". The "mathematical proof", in Section 2, shows that when there is a shield there generally exists a complete set of oscillatory modes; the "physical proof", in the last paragraphs of the article, shows that in practice the shield is unnecessary, since the energy leakage in its absence is negligible.